Pf (of derivative formula) $x_n = f(x_1, \dots, x_{n-1})$

Apply chain rule to compute partial $\partial F / \partial x_i$

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_i} + \frac{\partial F}{\partial x_n}\cdot\frac{\partial x_n}{\partial x_i}$$

$$0 \to \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_i}\cdot\cancel{\frac{\partial x_i}{\partial x_i}}^{1} + \frac{\partial F}{\partial x_n}\cdot\frac{\partial f}{\partial x_i} \to \frac{\partial f}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial x_n}$$

Ex. Compute $x^3+y^3+z^3 = 6xyz+1$ $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Sol:

$x^3+y^3+z^3 = 6xyz+1$ iff $x^3+y^3+z^3-6xyz-1=0$

use $F(x,y,z) = x^3+y^3+z^3-6xyz-1$

$$\frac{\partial F}{\partial x} = 3x^2-6yz, \quad \frac{\partial F}{\partial y} = 3y^2-6xz, \quad \frac{\partial F}{\partial z} = 3z^2-6xy$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{3x^2-6yz}{3z^2-6xy}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = \frac{3y^2-6xz}{3z^2-6xy}$$

Gradient and optimization

Def: The gradient of a function $f(x_1, x_2, \dots, x_n)$ is

$$\nabla F = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_n} \right\rangle$$

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i} \qquad \vec{x} = \langle x_1(t_1, \dots, t_n), x_2(t_1, \dots, t_n), x_3(t_1, \dots, t_n) \rangle$$

$$\frac{\partial \vec{x}}{\partial t_i} = \left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \frac{\partial x_n}{\partial t_i} \right.$$

$D_u f(\vec{p}) = \lim_{h\to 0} \frac{f(\vec{p}+h\vec{u}) - f(\vec{p})}{h}$, $D_u f(\vec{p}) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0)$

Consider $g(h) = f(\vec{p} + h\vec{u})$

$g(h) = f(p_1 + hu_1, p_2 + hu_2, p_n + hu_n)$

$\frac{dg}{dh} = \nabla f \cdot \frac{\partial f}{\partial h}$

$= \nabla f \cdot \langle u_1, u_2, \dots u_n \rangle = \nabla f \cdot \vec{u}$

$D_u f(\vec{p}) = \nabla f_{(p)} \cdot \vec{u}$

Ex. Lets compute the $D_{\vec{u}} f(\vec{p})$ for $f(x,y,z) = 4y\sqrt{x}$ $p = \langle 4, 1 \rangle$

$\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\nabla f = \langle 2x^{-\frac{1}{2}}y, 4x^{\frac{1}{2}} \rangle$, $\nabla f_{(p)} = \langle 2(4)^{-\frac{1}{2}}(1), 4(\frac{1}{2})^{\frac{1}{2}} \rangle = \langle 1, 8 \rangle$

$D_u f(\vec{p}) = \langle 1, 8 \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = \frac{1}{\sqrt{2}} - \frac{8}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$

Ex. Compute ∇f for $f(x,y,z) = \frac{xz}{y+z}$

Sol:

$\frac{\partial f}{\partial x} = \frac{z}{y+z}$, $\frac{\partial f}{\partial y} = -\frac{xz}{(y+z)^2}$

$\frac{\partial f}{\partial z} = \frac{(x)(y+z) - (1)(xz)}{(y+z)^2} = \frac{xy}{(y+z)^2}$

$\nabla f = \langle \frac{z}{y+z}, -\frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \rangle$

Gradient optimizes direction derivative

$\nabla f(p)$ direction realizes the maximum D_u ~~and~~ of $D_u f(p)$

$$D_u f(p) = \nabla f(\vec{p}) \cdot \vec{u} = |\nabla f(p)||\vec{u}| \cos\theta = |\nabla f(p)| \cos\theta$$

Ex. In which direction is $f(x,y,z) = \frac{xz}{y+z}$ attain is maximal directional derivative $p = \langle 1, 1, -2 \rangle$? What is the max?

Sol: $D_u f(p)$ is maximized in the direction of $\nabla f(p)$

$$\nabla f = \left\langle \frac{z}{y+z}, -\frac{xz}{(y+z)^2}, \frac{xy}{(y+z)^2} \right\rangle$$

$$\nabla f(p) = \left\langle \frac{-2}{1-2}, -\frac{-2}{(1-2)^2}, \frac{1}{(1-2)^2} \right\rangle = \langle 2, 2, 1 \rangle$$

$D_u f(p)$ is maximized in direction $\vec{u} = \frac{1}{3}\langle 2, 2, 1 \rangle$ and maximum $|\nabla f(p)| = 3$

Def: Let f be a function f has a maximum at P when $f(\vec{p}) \geq f(\vec{x})$ for all x nearby to $\vec{p}$. F has a global maximum value at $\vec{p}$ when $f(\vec{p}) = f(\vec{x})$,

Def: A critical point of function F is a point $\vec{p}$ such that either $\nabla f(p)$ does not exist or $\nabla f(p) = \vec{0}$

If f attains a local extrema at p then p is a critical point